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The structure of the Stewartson layers in a gas centrifuge. Part 2. Insulated side wall

By TAKUYA MATSUDA† AND HIDENORI TAKEDA
Department of Aeronautical Engineering, Kyoto University, Kyoto, Japan

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The Stewartson $E^{\frac{1}{3}}$- and $E^{\frac{1}{4}}$-layers in a rapidly rotating compressible fluid are considered within the framework of linearized equations and the boundary-layer method. The fluid is contained in a cylinder made of a thermally insulated side wall and conducting top and bottom end plates. The end plates and the side wall rotate with slightly different angular velocities. The case of an incompressible fluid was discussed by Stewartson, who found that the flow is restricted to the side-wall boundary layers. In the case of a compressible fluid, however, the solutions are strongly dependent upon the thermal boundary conditions assumed on the side wall. In particular, if the wall is insulated the fluid in the inner core is dragged along too since it is coupled strongly to the flow in the side-wall Stewartson layers. The critical parameter governing the solutions is found to be $(\gamma-1)\,PrG_0\,E^{-\frac{1}{3}}/4\gamma$, where γ is the ratio of specific heats, Pr the Prandtl number, G_0 the square of the rotational Mach number and E the Ekman number.

1. Introduction

The dynamics of rotating fluids have been mainly developed in the field of geophysics (see, for example, Greenspan 1967). In this field Coriolis forces play a dominant role while centrifugal forces are negligible. Another application of rotating-fluid dynamics is the engineering device known as the gas centrifuge, which is used for the separation of uranium isotopes. In this device the centrifugal force is as much as a factor of 10^5 greater than gravity, and therefore cannot be neglected.

Thermally driven meridional flows in a rapidly rotating cylinder have been considered by Barcilon & Pedlosky (1967) for the case where the centrifugal force is of the same order as gravity, while Homsy & Hudson (1969) studied the case where the centrifugal force is much greater than gravity on the basis of the Boussinesq approximation. The Boussinesq approximation, however, is inappropriate for describing a gas centrifuge, since the peripheral speed of the gas is as large as 400 m/s and consequently the rotational Mach number exceeds unity. In view of this fact, several workers, including the present authors, have developed the theory of rapidly rotating compressible fluids (Sakurai & Matsuda 1974; Matsuda, Sakurai & Takeda; 1975 Matsuda, Hashimoto & Takeda 1976; Matsuda & Hashimoto 1976; these papers will be referred to as I, II, III and IV respectively). The theory has been applied to practical gas centrifuges by Nakayama & Usui (1974) and by Matsuda (1975, 1976*a*). Recently Durivault &

† Temporary Address: Department of Applied Mathematics and Astronomy, University College, Cardiff, Wales.

Louvet (1976*a*) derived similar solutions and obtained essentially the same result as that in I.

In I–IV, the authors avoided the appearance of a Stewartson $E^{\frac{1}{4}}$-layer by imposing appropriate boundary conditions, but this was only for the sake of simplicity. However, an $E^{\frac{1}{4}}$-layer must be present in a gas centrifuge owing to the presence of narrow source and/or sink slits (II) or scoops for collecting gas. Recently Matsuda (1976*b*) proposed a gas centrifuge in which the end plates rotate faster than the side wall in order to improve the efficiency of the centrifuge. In such a device the appearance of an $E^{\frac{1}{4}}$-layer is inevitable. In view of these facts, it is important to investigate the structure of a Stewartson $E^{\frac{1}{4}}$-layer as well as the $E^{\frac{1}{3}}$-layer in a rapidly rotating compressible fluid.

Nakayama & Usui (1974) considered the Stewartson layers by assuming the density scale height to be larger than the thickness of these layers and by expanding the equations in terms of the ratio of the thickness to the scale height. While Durivault & Louvet (1976*b*) solved the $E^{\frac{1}{3}}$-layer numerically, Bark & Bark (1976) found the exact solution for the $E^{\frac{1}{4}}$-layer and asymptotic solutions for the $E^{\frac{1}{3}}$-layer as well as numerical solutions. However these authors assumed that the cylinder walls are thermally conducting, i.e. that the temperature of the wall is specified.

In III and IV we stressed the important effect of the thermal boundary condition on the inner flow field, particularly if the walls are thermally insulated. The combined effect of the compressibility of the fluid and the thermal insulation of the walls produces a flow distinctly different from that of Boussinesq fluids and from that of compressible fluids confined in a cylinder with thermally conducting walls. Therefore it is an interesting fluid-dynamic problem in itself to consider the case of thermally insulated walls, as well as being of practical importance for gas centrifuges.

In the present paper (and in the previous paper, Matsuda & Hashimoto 1978, referred to as part 1) we assume a simple configuration in which the top and bottom end plates rotate with angular velocity Ω, while the side wall rotates with a slightly different velocity $\Omega+\Delta\Omega$. There are assumed to be no source–sink or temperature distributions on the end plates and we consider the case of a thermally insulated side wall and conducting end plates. The opposite case was considered in part 1.

Before discussing the complicated mathematical details, we briefly consider the physical nature of the problem. As a first step we consider the case of a Boussinesq (incompressible) fluid. Figure 1 shows the flow field for this case in the r, z plane of the cylinder. The flow is divided into several regions: the inner inviscid core, the Stewartson $E^{\frac{1}{4}}$-layer along the side wall, its extension to the top and bottom Ekman layers with thickness $E^{\frac{1}{2}}$, the Stewartson $E^{\frac{1}{3}}$-layer and its extension to the Ekman layers. The two $E^{\frac{1}{2}}\times E^{\frac{1}{2}}$ square regions in the corners will be ignored in the following since they are unimportant (shaded areas in figure 1). In a frame rotating with angular velocity Ω, there is no flow in the inner core on account of the Taylor–Proudman theorem. Along the side wall, however, there must be azimuthal flow of order unity so as to satisfy the side-wall conditions. (Here we non-dimensionalize the quantities such that the dimensionless $\Delta\Omega$ becomes of order unity.) This azimuthal flow does not satisfy the boundary conditions at the top and bottom end plates, leading to the formation of the top and bottom Ekman layers with thickness $E^{\frac{1}{2}}$, which in turn induce a meridional flow of order $E^{\frac{1}{2}}$ in the $E^{\frac{1}{4}}$-layer owing to Ekman suction. This flow circulates through the $E^{\frac{1}{3}}$-layer and the Ekman extensions as shown in figure 1. In this case there is no

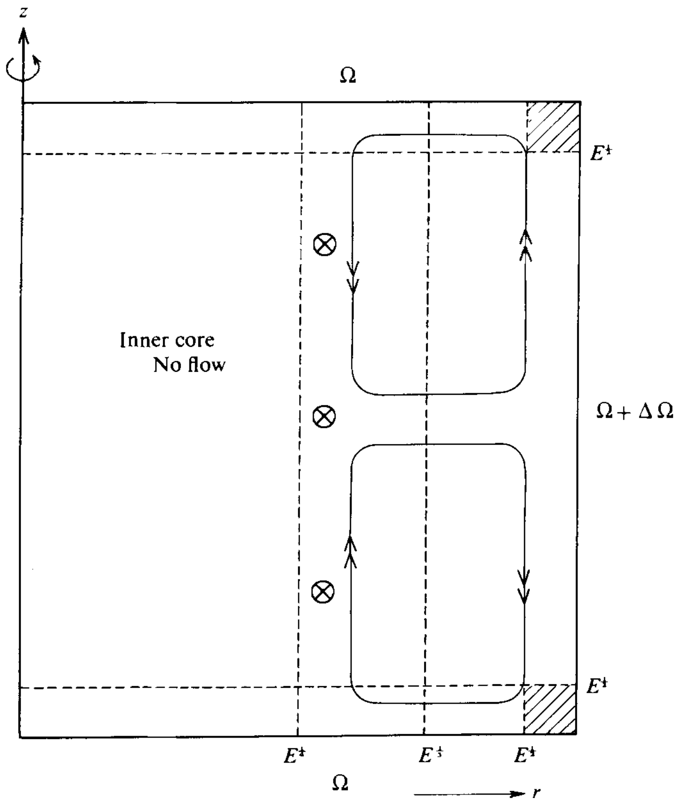

FIGURE 1. Schematic diagram of the flow field of an incompressible fluid in the r, z plane.

closed circulation of magnitude $O(E^{\frac{1}{3}})$ in the $E^{\frac{1}{3}}$-layer, as would appear if there were a z-dependent azimuthal flow in the inner core (Hunter 1967).

Now we turn to the case of a compressible fluid. As was stressed in III and IV, radial motion of fluid elements generates or absorbs heat via the work done by pressure forces. This heat supply or loss is compensated for by conduction through the side wall if it is conducting, but not in the case of an insulated wall. As radial motion mainly occurs along the wall, the coupled effect of compressibility and an insulated wall is drastic when the rotational velocity of the cylinder is so high that the rotational Mach number becomes of order unity. Figure 2 shows the flow field of a compressible fluid for a case similar to that in figure 1. The mechanism producing the meridional circulation $O(E^{\frac{1}{2}})$ through the $E^{\frac{1}{4}}$- and $E^{\frac{1}{3}}$-layers is the same as in the case in figure 1. However the outward motion of the circulation about the midplane generates heat, which penetrates into the inner core and warms the fluid. The induced increase in temperature in turn produces an azimuthal thermal wind in the inner core. As this thermal wind should be z dependent, a closed circulation in the $E^{\frac{1}{3}}$-layer is set up in this case in order to satisfy the side-wall boundary conditions. This closed circulation generates further heat owing to the associated radial motion, and the inner temperature field is further affected. The flow and the temperature field are determined by all these processes in a self-consistent manner. The most striking phenomenon is, therefore, that the inner fluid is dragged slightly in the direction of the velocity of the side wall.

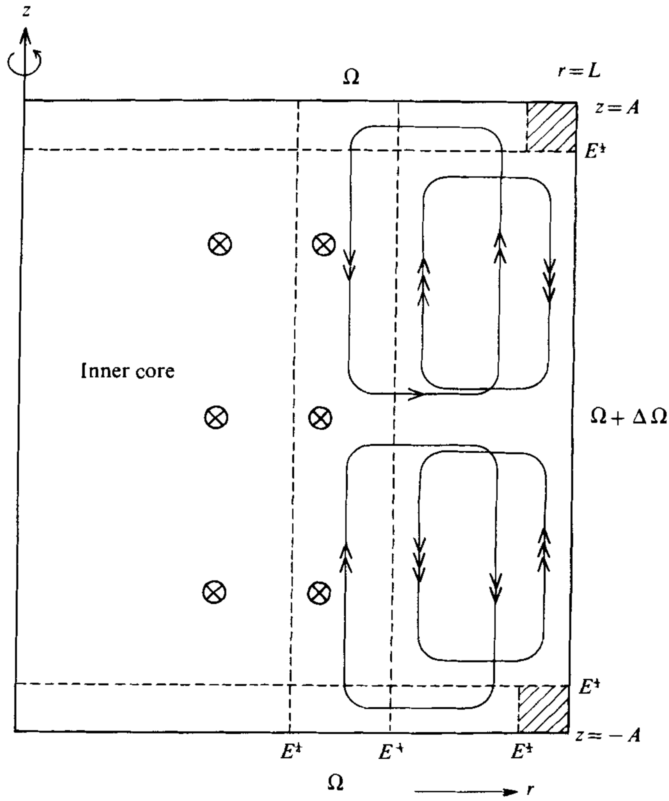

FIGURE 2. Schematic diagram of the flow field of a compressible fluid in the r, z plane.

In § 2 the linearized basic equations and boundary conditions are given. In § 3 we give a formal solution for the flow in the inner core and the boundary layers. In § 4 the equations derived in § 3 are reduced to an infinite set of linear equations by an expansion method, and an approximate solution for these is obtained. Finally, we discuss the results in § 5.

2. Formulation

A compressible fluid, such as UF_6, is confined in a cylinder of radius L and height $2H$ rotating about its vertical axis. The rotational angular velocity of both the top and the bottom end plate is Ω, while that of the side wall is $\Omega + \Delta\Omega$. The temperature of the end plates is kept constant, say at $\tilde{T}_0$, while the side wall is thermally insulated. Ω is assumed to be so high that gravity is negligible compared with the characteristic centrifugal acceleration $\Omega^2 L$. The peripheral speed of the cylinder ΩL is assumed to be of the order of the sound speed of the working fluid, therefore we must take into account the compressibility of the fluid.

We shall linearize the equations with respect to the Rossby number

$$\delta = |\Delta\Omega|/\Omega; \tag{2.1}$$

then in the zero-order state the gas rotates rigidly with the end plates and the distributions of the density and the pressure are determined by a balance between the centrifugal force and the pressure gradient:

$$\tilde{p}_R = \tilde{p}_2 \epsilon_R, \quad \tilde{\rho}_R = M\tilde{p}_R/R\tilde{T}_0, \tag{2.2}$$

$$\epsilon_R = \exp\left[M\Omega^2(\tilde{r}^2 - L^2)/2R\tilde{T}_0\right], \tag{2.3}$$

where M and R are respectively the mean molecular weight of the gas and the universal gas constant, $\tilde{r}$ is the radial distance, $\tilde{p}_2$ the pressure at $\tilde{r} = L$ and the tildes denote dimensional quantities. Before linearizing the equations, we define the following non-dimensional quantities:

$$\left.\begin{aligned} (r, z) &= (\tilde{r}/L, \tilde{z}/L), \quad (u, v, w) = (\tilde{q}_r, \tilde{q}_\theta, \tilde{q}_z)/L\Omega\delta, \\ T &= (\tilde{T} - \tilde{T}_0)/G_0\tilde{T}_0\delta, \quad p = (\tilde{p} - \tilde{p}_R)/\tilde{p}_R\delta, \end{aligned}\right\} \tag{2.4}$$

where $(\tilde{q}_r, \tilde{q}_\theta, \tilde{q}_z)$ are dimensional velocity components in the rotating frame and G_0 will be defined in (2.11). The ranges of r and z are $0 \leqslant r \leqslant 1$ and $-A \leqslant z \leqslant A$, where $A = H/L$.

Assuming δ to be small and neglecting nonlinear terms, we then obtain the following linearized equations:

$$\operatorname{div} \mathbf{q} + G_0 r u = 0, \tag{2.5}$$

$$-2v + rT + G_0^{-1}\partial p/\partial r = (E/\epsilon_R)\left[\mathscr{L}u + \tfrac{1}{3}\partial(\operatorname{div}\mathbf{q})/\partial r\right], \tag{2.6}$$

$$2u = (E/\epsilon_R)\,\mathscr{L}v, \tag{2.7}$$

$$G_0^{-1}\partial p/\partial z = (E/\epsilon_R)\left[\Delta w + \tfrac{1}{3}\partial(\operatorname{div}\mathbf{q})/\partial z\right], \tag{2.8}$$

$$-4hru = (E/\epsilon_R)\,\Delta T, \tag{2.9}$$

$$\operatorname{div}\mathbf{q} = r^{-1}\partial(ru)/\partial r + \partial w/\partial z, \quad \mathscr{L} = \Delta - r^{-2}, \quad \Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2, \tag{2.10}$$

$$G_0 = M(L\Omega)^2/R\tilde{T}_0, \quad E = \mu/\tilde{p}_2\,\Omega L^2, \quad h = (\gamma - 1)\,PrG_0/4\gamma, \quad \epsilon_R = \exp\left[\tfrac{1}{2}G_0(r^2 - 1)\right]. \tag{2.11}$$

Here Pr is the Prandtl number, γ the ratio of specific heats, μ the viscosity coefficient, G_0 the square of the rotational Mach number, E the Ekman number and h a parameter representing the hardness (or compressibility) of gas. As γ is very close to unity for polyatomic molecules such as UF_6 ($\gamma = 1{\cdot}067$ for UF_6), the order of magnitude of h is 10^{-2} even if G_0 is $O(1)$. In a gas centrifuge the order of E is $10^{-6} \sim 10^{-7}$ while G_0 is about 20, but we assume G_0 to be $O(1)$ for the sake of simplicity. The boundary conditions applied to (2.5)–(2.9) are

$$u = v = w = 0, \quad T = 0 \quad \text{on} \quad z = \pm A, \tag{2.12}$$

$$u = w = 0, \quad v = 1, \quad \partial T/\partial r = 0 \quad \text{on} \quad r = 1. \tag{2.13}$$

3. Boundary-layer analysis

Since we have neglected gravity, the boundary layers appearing near the side wall are the Stewartson $E^{\frac{1}{3}}$- and $E^{\frac{1}{4}}$-layers (Stewartson 1957), and other layers such as a buoyancy layer (Barcilon & Pedlosky 1967) do not appear. In addition, since there exists an azimuthal wind in these layers, there must be Ekman layers at their top and bottom ends. The interior of the cylinder may be divided into the following regions: the inner inviscid core, the $E^{\frac{1}{4}}$-layer, its Ekman extensions, the $E^{\frac{1}{3}}$-layer and its Ekman extensions (see figure 1).

Inner inviscid core

In the inner inviscid core we can neglect viscosity, therefore (2.6)–(2.8) reduce to

$$v_i = \tfrac{1}{2}(rT_i - G_0^{-1}\,\partial p_i/\partial r), \quad u_i = 0, \quad \partial p_i/\partial z = 0,$$

where the suffix i denotes a value in the inner inviscid core. As can be seen, p_i is a function of r only, and since we have assumed the absence of any sources or sinks, $v_i(z = \pm A) = 0$. Noting that $T_i(z = \pm A) = 0$, we can take p_i to be identically zero. Therefore we obtain the following thermal-wind relation:

$$v_i = \tfrac{1}{2}rT_i. \tag{3.1}$$

$E^{\frac{1}{4}}$-layer

In order to analyse the $E^{\frac{1}{4}}$-layer, we introduce the following scaling:

$$\left.\begin{aligned} u = E^{\frac{1}{2}}\tilde{u}, \quad v = v_i + \tilde{v}, \quad w = E^{\frac{1}{4}}\tilde{w}, \\ p = G_0(1+h)\,E^{\frac{1}{4}}\tilde{p}, \quad T = T_i + h\tilde{T}, \quad \xi = (1-r)/E^{\frac{1}{4}}, \end{aligned}\right\} \tag{3.2}$$

where quantities with a tilde† are boundary-layer components of order unity, ξ is a stretched radial co-ordinate and higher-order quantities are neglected.‡ Inserting (3.2) into (2.5)–(2.9) and using the inner-core solutions, we obtain

$$\partial\tilde{u}/\partial\xi - \partial\tilde{w}/\partial z = 0, \tag{3.3}$$

$$-2\tilde{v} + h\tilde{T} - (1+h)\,\partial\tilde{p}/\partial\xi = 0, \tag{3.4}$$

$$2\tilde{u} = \partial^2\tilde{v}/\partial\xi^2, \tag{3.5}$$

$$\partial\tilde{p}/\partial z = 0, \quad -4\tilde{u} = \partial^2\tilde{T}/\partial\xi^2, \qquad (3.6), (3.7)$$

where we have put $r = 1$. The major error terms are $O(G_0\,E^{\frac{1}{4}})$ and $O(E^{\frac{1}{4}})$. In a practical gas centrifuge the terms $O(G_0\,E^{\frac{1}{4}})$ are the larger since G_0 is greater than unity. In view of this, Nakayama & Usui (1974) expanded the solutions in powers of $G_0\,E^{\frac{1}{4}}$, while Bark & Bark (1976) obtained the exact solution including the term of order $G_0\,E^{\frac{1}{4}}$. However, in the present analysis, we neglect the terms $O(G_0\,E^{\frac{1}{4}})$ as well as those $O(E^{\frac{1}{4}})$. If we assume the side wall to be thermally insulated, the parameter $\alpha\,(\equiv hE^{-\frac{1}{3}})$ introduced in III appears in the problem and since the terms including α make the analysis very complicated compared with the case of a conducting side wall, we wish to avoid the additional complexity associated with the terms of order $G_0\,E^{\frac{1}{4}}$. In fact, in a practical gas centrifuge, $G_0\,E^{\frac{1}{4}}$ is estimated to be at most $O(1)$ while α is $O(10)$. Therefore the effect of the α terms is more important than that of the $G_0\,E^{\frac{1}{4}}$ terms.

Eliminating $\tilde{u}$ from (3.5) and (3.7) and noting that $\tilde{v}$, $\tilde{T} \to 0$ as $\xi \to \infty$, we obtain

$$\tilde{v} = -\tfrac{1}{2}\tilde{T}. \tag{3.8}$$

† Not to be confused with the original dimensional quantities.

‡ u_i and w_i are omitted because they are smaller than $\tilde{u}$ and $\tilde{w}$.

As the pressure is a function of ξ only, the following relations are obtained from (3.4) and (3.8):

$$\tilde{v} = \tfrac{1}{2}f(\xi), \quad \tilde{T} = -f(\xi), \tag{3.9}$$

where $f(\xi) \equiv -\partial\tilde{p}/\partial\xi$. Insertion of (3.9) into (3.5) gives

$$\tilde{u} = \tfrac{1}{4}f''(\xi), \tag{3.10}$$

where a prime denotes differentiation with respect to ξ. Inserting (3.10) into (3.3) and integrating with respect to z, we obtain

$$\tilde{w} = \tfrac{1}{4}zf'''(\xi) + g(\xi), \tag{3.11}$$

where $g(\xi)$ is an unknown function to be determined later. As can easily be seen from the scaling (3.2), the mass flux in the $E^{\frac{1}{4}}$-layer is $O(E^{\frac{1}{2}})$.

Ekman extension of $E^{\frac{1}{4}}$-layer

In order to determine the unknown functions $f(\xi)$ and $g(\xi)$, we need to consider the Ekman extension of the $E^{\frac{1}{4}}$-layer, since the meridional flow in the $E^{\frac{1}{4}}$-layer is induced by Ekman suction. Let us scale the variables in this region as follows:

$$\left.\begin{aligned} u = \hat{u}, \quad v = v_i + \tilde{v} + \hat{v}, \quad w = E^{\frac{1}{4}}(\tilde{w} + \hat{w}), \\ T = T_i + h(\tilde{T} + \hat{T}), \quad p = G_0(1+h)\,E^{\frac{1}{4}}(\tilde{p} + \hat{p}), \quad \eta = (A \mp z)/E^{\frac{1}{2}}, \end{aligned}\right\} \tag{3.12}$$

where carets refer to variables in the Ekman layers, η is a stretched co-ordinate and $\pm$ denote the top and bottom respectively. Substituting (3.12) into the basic equations, we obtain the following solutions:

$$\hat{T} = D_{\pm}(\xi)\,e^{-\sigma\eta}\cos\sigma\eta, \quad \tilde{v} = -\tfrac{1}{2}D_{\pm}(\xi)\,e^{-\sigma\eta}\cos\sigma\eta, \tag{3.13}$$

$$\hat{u} = -\tfrac{1}{2}(1+h)^{\frac{1}{2}}D_{\pm}(\xi)\,e^{-\sigma\eta}\sin\sigma\eta, \quad \hat{w} = \mp\tfrac{1}{4}(1+h)^{\frac{1}{4}}D_{\pm}'(\xi)\,e^{-\sigma\eta}(\cos\sigma\eta + \sin\sigma\eta), \tag{3.14}$$

where $\sigma = [(1+h)\,\epsilon_R^2]^{\frac{1}{4}}$, and $D_{\pm}(\xi)$ will be determined later. Since $T(z = \pm A) = 0$, we obtain

$$0 = \tilde{T} + \hat{T} = -f(\xi) + D_{\pm}(\xi) \quad \text{on} \quad \eta = 0, \quad \text{i.e.} \quad z = \pm A, \tag{3.15}$$

where we have imposed the conditions $T_i(r = 1, z = \pm A) = 0$, which will be shown to be consistent. The boundary conditions on v are automatically satisfied by (3.15). The velocity w satisfies the boundary conditions

$$0 = \tilde{w} + \hat{w} = \pm\tfrac{1}{4}Af'''(\xi) + g(\xi) \mp \tfrac{1}{4}(1+h)^{\frac{1}{4}}D_{\pm}'(\xi). \tag{3.16}$$

In order that (3.16) holds at the top and bottom, $g(\xi)$ should be zero and therefore

$$Af'''(\xi) - (1+h)^{\frac{1}{4}}f'(\xi) = 0, \tag{3.17}$$

from (3.15). Since $f(\xi)$ and $f'(\xi)$ must approach zero as $\xi \to \infty$, we can easily integrate (3.17) to obtain

$$f(\xi) = c\exp\left[-A^{-\frac{1}{2}}(1+h)^{\frac{1}{8}}\,\xi\right], \tag{3.18}$$

where the constant c will be determined later.

$E^{\frac{1}{3}}$-layer

The scaling of physical quantities in the $E^{\frac{1}{3}}$-layer is

$$\left.\begin{aligned} u = E^{\frac{1}{3}}\bar{u}, \quad v = v_i + \bar{v}, \quad w = \bar{w}, \\ p = G_0 E^{\frac{1}{3}}\bar{p}, \quad T = T_i + h\bar{T}, \quad \zeta = (1-r)/E^{\frac{1}{3}}, \end{aligned}\right\} \tag{3.19}$$

where the bars denote variables in the $E^{\frac{1}{3}}$ boundary layer, which match onto the $E^{\frac{1}{4}}$-layer variables at the interface between these two layers.

Substituting (3.19) into the basic equations and neglecting terms of order $G_0 E^{\frac{1}{3}}$, $E^{\frac{1}{3}}$ and higher, we obtain the following equations:

$$\partial\bar{u}/\partial\zeta - \partial\bar{w}/\partial z = 0, \tag{3.20}$$

$$-2\bar{v} + h\bar{T} - \partial\bar{p}/\partial\zeta = 0, \tag{3.21}$$

$$2\bar{u} = \partial^2\bar{v}/\partial\zeta^2, \tag{3.22}$$

$$\partial\bar{p}/\partial z = \partial^2\bar{w}/\partial\zeta^2, \quad -4\bar{u} = \partial^2\bar{T}/\partial\zeta^2. \qquad (3.23), (3.24)$$

Eliminating $\bar{u}$ from (3.22) and (3.24) and integrating the resultant equations, we obtain

$$2\bar{v} + \bar{T} = f_1(z) + f_2(z)\,\zeta.$$

Matching of the solutions for the $E^{\frac{1}{3}}$-layer with those for the $E^{\frac{1}{4}}$-layer gives

$$\lim_{\zeta\to\infty}(2\bar{v} + \bar{T}) = \lim_{\xi\to 0}(2\tilde{v} + \tilde{T}).$$

The right-hand side is identically zero [see (3.8)], consequently $f_1(z)$ and $f_2(z)$ must be zero. Therefore we obtain

$$\bar{v} = -\tfrac{1}{2}\bar{T}. \tag{3.25}$$

Elimination of all dependent variables except $\bar{T}$ leads to the following governing equation for $\bar{T}$:

$$\partial^6\bar{T}/\partial\zeta^6 + 4(1+h)\,\partial^2\bar{T}/\partial z^2 = 0. \tag{3.26}$$

If a stream function $\bar{\psi}$ defined by

$$\bar{u} = \partial\bar{\psi}/\partial z, \quad \bar{w} = \partial\bar{\psi}/\partial\zeta \tag{3.27}$$

is introduced then the boundary conditions for $\bar{\psi}$ are $\bar{\psi}(z = \pm A) = 0$. Substituting (3.25) and (3.27) into (3.22) and integrating the resultant equation with respect to z we obtain

$$\int_{-A}^{A} (\partial^2\bar{T}/\partial\zeta^2)\,dz = 0, \tag{3.28}$$

where the boundary conditions for $\bar{\psi}$ have been used. Integrating (3.28) twice with respect to ζ, we obtain

$$\int_{-A}^{A} \bar{T}\,dz = 2A(a + b\zeta), \tag{3.29}$$

where the constants a and b will be determined later. The temperature $\bar{T}$ should consist of two parts: $a + b\zeta$, which is independent of z, and a part which satisfies the equation

$$\int_{-A}^{A} \bar{T}\,dz = 0.$$

Because of the symmetry of $\overline{T}$ as a function of z, we write $\overline{T}$ as a Fourier series:

$$\overline{T} = a + b\zeta + \sum_{m=1}^{\infty} \{f_{0m}(\zeta) + E^{\frac{1}{12}} f_{1m}(\zeta)\} \cos [m\pi(z+A)/A]. \tag{3.30}$$

Here the terms higher than $O(E^{\frac{1}{12}})$ have been neglected for the sake of simplicity. Within this approximation, the analysis of the Ekman extension of the $E^{\frac{1}{3}}$-layer is not required. The higher-order terms neglected represent a rechannelling flow $O(E^{\frac{1}{2}})$, while the lower-order terms represent a closed circulation $O(E^{\frac{1}{3}})$ and $O(E^{\frac{5}{12}})$ in the $E^{\frac{1}{3}}$-layer. Since $\tilde{T}$ does not depend on z, it follows that $f_{0m}(\zeta)$ and $f_{1m}(\zeta)$ must vanish as $\zeta \rightarrow \infty$, and these conditions ensure that $\overline{u}, \overline{w} \rightarrow 0$ as $\zeta \rightarrow \infty$.

The side-wall boundary conditions (2.13) reduce to

$$\left.\begin{aligned} \overline{u} = \overline{w} &= 0 \qquad &(3.31)\\ v_i + \overline{v} = 1, \quad \partial T_i/\partial r - hE^{-\frac{1}{3}} \partial \overline{T}/\partial \zeta &= 0 \qquad &(3.32) \end{aligned}\right\} \text{ on } r = 1, \text{ i.e. } \zeta = 0.$$

By using (3.20), (3.22) and (3.25) the conditions (3.31) can be re-written in terms of $f_{im}(\zeta)$ $(i = 0, 1)$ as follows:

$$f''_{im}(0) = f''''_{im}(0) = 0, \quad i = 0, 1. \tag{3.33}$$

The solutions of (3.26) satisfying (3.33) are

$$f_{im}(\zeta) = \tfrac{1}{2} f_{im}(0) \left[\exp(-\omega_m \zeta) + \frac{2}{3^{\frac{1}{2}}} \exp(-\tfrac{1}{2}\omega_m \zeta) \cos\left(\frac{3^{\frac{1}{2}}}{2}\omega_m \zeta - \frac{\pi}{6}\right)\right], \quad i = 0, 1, \tag{3.34}$$

where $\omega_m = [2m(1+h)^{\frac{1}{2}}/A]^{\frac{1}{3}}$, and the $f_{im}(0)$ are unknown constants to be determined later. Using (3.1), (3.25) and (3.30), the boundary conditions (3.32) can be re-written in terms of the temperature:

$$T_i - \left\{a + \sum_{m=1}^{\infty} [f_{0m}(0) + E^{\frac{1}{12}} f_{1m}(0)] \cos [\gamma_m(z+A)]\right\} = 2, \tag{3.35}$$

$$\partial T_i/\partial r - \alpha \left\{b + \sum_{m=1}^{\infty} [f'_{0m}(0) + E^{\frac{1}{12}} f'_{1m}(0)] \cos [\gamma_m(z+A)]\right\} = 0, \tag{3.36}$$

with $\alpha = hE^{-\frac{1}{3}}$.

Matching of solutions

In order to match the solutions for the $E^{\frac{1}{3}}$- and $E^{\frac{1}{4}}$-layers at their interface, it is convenient to expand the constants a, b and c in powers of $E^{\frac{1}{12}}$. The series must be in ascending powers of $E^{\frac{1}{12}}$ (Hunter 1967), i.e.

$$a = a_0 + a_1 E^{\frac{1}{12}} + \ldots, \quad b = b_0 + b_1 E^{\frac{1}{12}} + \ldots, \quad c = c_0 + c_1 E^{\frac{1}{12}} + \ldots. \tag{3.37}$$

From a Taylor expansion of $\tilde{T}$ about $\xi = 0$, we obtain

$$\begin{aligned} \overline{T} &= -c \exp[-A^{-\frac{1}{2}}(1+h)^{\frac{1}{6}} \xi] = -(c_0 + c_1 E^{\frac{1}{12}} + \ldots)[1 - A^{-\frac{1}{2}}(1+h)^{\frac{1}{6}} \xi + \ldots] \\ &= -[c_0 + c_1 E^{\frac{1}{12}} - c_0 A^{-\frac{1}{2}} (1+h)^{\frac{1}{6}} E^{\frac{1}{12}} \zeta + O(E^{\frac{1}{6}})]. \end{aligned} \tag{3.38}$$

Comparing (3.38) with $a_0 + a_1 E^{\frac{1}{12}} + (b_0 + b_1 E^{\frac{1}{12}}) + O(E^{\frac{1}{6}})$, which is an asymptotic form of $\overline{T}$ as $\zeta \rightarrow \infty$ from (3.30), we obtain

$$a_0 = -c_0, \quad a_1 = -c_1, \quad b_0 = 0, \quad b_1 = c_0(1+h)^{\frac{1}{6}} A^{-\frac{1}{2}}. \tag{3.39}$$

4. Solution for the inner temperature field

The whole problem has been reduced to finding a solution for T_i. The equation governing T_i is derived from (2.7), (2.9) and (3.1) and has the form

$$\Delta T_i + \tfrac{1}{2} hr \mathscr{L}(rT_i) = 0. \tag{4.1}$$

Hereafter we neglect h in the same manner as in III and IV; then (4.1) reduces to

$$\Delta T_i = 0. \tag{4.2}$$

The boundary conditions for (4.2) are (3.35), (3.36) and $T_i(z = \pm A) = 0$, together with (3.34) and (3.39).

Careful inspection of (3.35) and (3.36) gives us the order of magnitude of T_i. As the right-hand side of (3.35) is $O(1)$, it is easily seen that a is also $O(1)$. Therefore the leading term of b is $O(E^{\frac{1}{12}})$ and that of c is $O(1)$. If we assume that α is $O(1)$, inspection of (3.36) shows that T_i is $O(E^{\frac{1}{12}})$. Before obtaining the solution for T_i, however, it may not be completely obvious that T_i does not have terms $O(1)$. Therefore for the sake of completeness we expand T_i in the following manner:

$$T_i = \sum_{n=1}^{\infty} [g_{0n}(r) + E^{\frac{1}{12}} g_{1n}(r)] \sin[\beta_n(z+A)], \tag{4.3}$$

where $\beta_n = (2n-1)/2A$ and terms higher than $O(E^{\frac{1}{12}})$ are neglected. This Fourier sine series satisfies automatically the boundary conditions at $z = \pm A$ and has the correct symmetry for T_i.

Substituting (4.3) into (4.2) and requiring T_i to be non-singular at $r = 0$, we easily obtain

$$g_{in}(r) = a_{in} I_0(\beta_n r), \tag{4.4}$$

where I_0 is a modified Bessel function of order zero, and the constants a_{in} are determined by the boundary conditions.

Substitution of (4.3) and (4.4) into (3.35) gives

$$\sum_{n=1}^{\infty} a_{0n} I_0(\beta_n) \sin[\beta_n(z+A)] - \sum_{m=1}^{\infty} f_{0m}(0) \cos[\gamma_m(z+A)] = 2 - c_0 \tag{4.5}$$

and

$$\sum_{n=1}^{\infty} a_{1n} I_0(\beta_n) \sin[\beta_n(z+A)] - \sum_{m=1}^{\infty} f_{1m}(0) \cos[\gamma_m(z+A)] = -c_1. \tag{4.6}$$

A similar procedure for (3.36) gives

$$\sum_{n=1}^{\infty} a_{0n} \beta_n I_1(\beta_n) \sin[\beta_n(z+A)] + \frac{\alpha}{2} \sum_{m=1}^{\infty} \omega_m f_{0m}(0) \cos[\gamma_m(z+A)] = 0, \tag{4.7}$$

$$\sum_{n=1}^{\infty} a_{1n} \beta_n I_1(\beta_n) \sin[\beta_n(z+A)] + \frac{\alpha}{2} \sum_{m=1}^{\infty} \omega_m f_{1m}(0) \cos[\gamma_m(z+A)] = \alpha c_0 A^{-\frac{1}{2}}, \tag{4.8}$$

where I_1 is a modified Bessel function of order 1. Equations (4.5) and (4.7) determine the zeroth-order solutions in terms of $E^{\frac{1}{12}}$ while (4.6) and (4.8) give the first-order solutions. There are two methods of solving these equations; one is to expand

$$\sin[\beta_n(z+A)]$$

in terms of $\cos[\gamma_m(z+A)]$ and the other is to expand the cosines in sine series (see III). Here we show only the formulae for the former case, though both methods have been tried: both numerical calculations give the same results. Multiplying (4.5) and (4.7) by $\cos[\gamma_m(z+A)]$ and integrating from $-A$ to A, we obtain

$$\sum_{n=1}^{\infty} a_{0n} I_0(\beta_n) P_{mn}/\beta_n - A f_{0m}(0) = 0, \tag{4.9}$$

$$\sum_{n=1}^{\infty} a_{0n} I_1(\beta_n) P_{mn} + \frac{\alpha}{2} A\omega_m f_{0m}(0) = 0, \tag{4.10}$$

where $P_{mn} = 2/\{1-[2m/(2n-1)]^2\}$.

Integrating (4.5) and (4.7) as they stand from $-A$ to A, we obtain

$$\sum_{n=1}^{\infty} a_{0n} I_0(\beta_n)/\beta_n = A(2-c_0), \tag{4.11}$$

$$\sum_{n=1}^{\infty} a_{0n} I_1(\beta_n) = 0. \tag{4.12}$$

Eliminating $f_{0m}(0)$ from (4.9) and (4.10) and solving the resulting infinite linear algebraic equations for a_{0n} together with (4.12), we obtain

$$a_{0n} = 0. \tag{4.13}$$

Equations (4.13), (4.10) and (4.11) give

$$c_0 = 2, \quad f_{0m}(0) = 0. \qquad (4.14), (4.15)$$

Therefore it can be concluded that there is no inner temperature field $O(1)$ nor a closed circulation $O(E^{\frac{1}{3}})$ in the $E^{\frac{1}{4}}$-layer as was expected. The $E^{\frac{1}{4}}$-layer is the same as for the case of a Boussinesq fluid as far as the lowest order is concerned.

The first-order equations are derived from (4.6) and (4.8) in a similar manner to (4.9) etc.:

$$\sum_{n=1}^{\infty} a_{1n} I_0(\beta_n) P_{mn}/\beta_n - A f_{1m}(0) = 0, \tag{4.16}$$

$$\sum_{n=1}^{\infty} a_{1n} I_1(\beta_n) P_{mn} + \frac{\alpha}{2} A\omega_m f_{1m}(0) = 0, \tag{4.17}$$

$$\sum_{n=1}^{\infty} a_{1n} I_0(\beta_n)/\beta_n = -Ac_1 = Aa_1, \tag{4.18}$$

$$\sum_{n=1}^{\infty} a_{1n} I_1(\beta_n) = A^{\frac{1}{2}}\alpha c_0. \tag{4.19}$$

Eliminating $f_{1m}(0)$ from (4.16) and (4.17), we obtain

$$\sum_{n=1}^{\infty} [\alpha\omega_m I_0(\beta_n)/\beta_n + 2I_1(\beta_n)] P_{mn} a_{1n} = 0. \tag{4.20}$$

Then, combining (4.20) with (4.19) and solving the resulting infinite linear algebraic equations, we obtain a_{1n}. Substitution of a_{1n} into (4.16) and (4.18) gives $f_{1m}(0)$ and c_1 (or a_1).

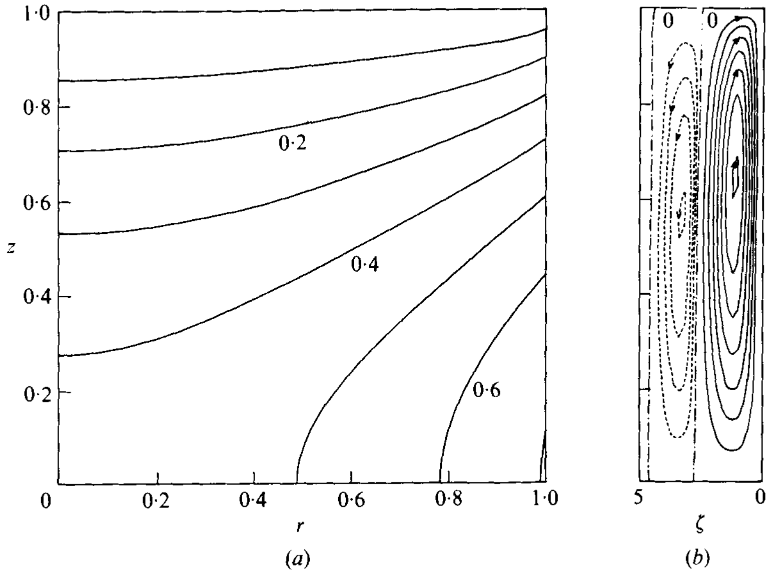

FIGURE 3. (*a*) Isotherms in the inner core. (*b*) Streamlines in the side-wall Stewartson $E^{\frac{1}{3}}$-layer. (Only the upper half of the cylinder is shown.) The difference in the values of the non-dimensional flux on adjacent lines is $-0{\cdot}002$ for solid lines and $0{\cdot}0004$ for dashed lines. Aspect ratio $A = 1$, $\alpha = hE^{-\frac{1}{3}} = 0{\cdot}3$.

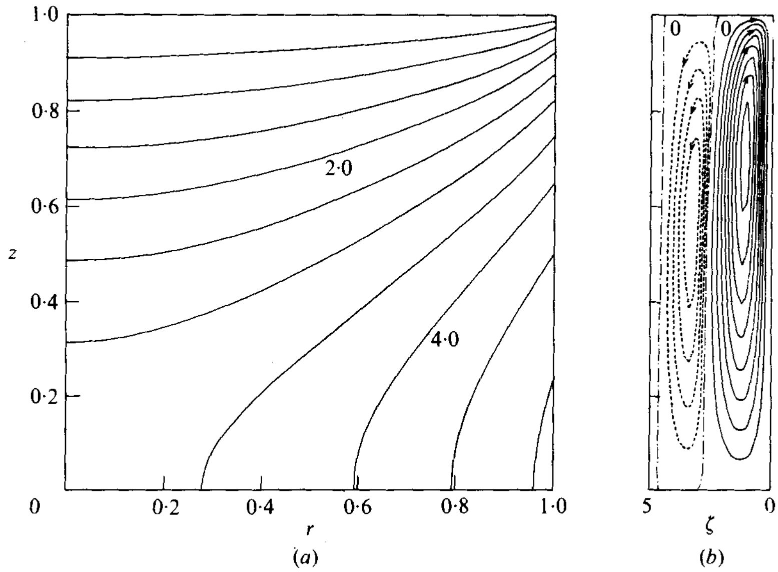

FIGURE 4. (*a*) Isotherms in the inner core. (*b*) Streamlines in the side-wall Stewartson $E^{\frac{1}{3}}$-layer. The difference in the values of the non-dimensional flux on adjacent lines is $-0{\cdot}01$ for solid lines and $0{\cdot}002$ for dashed lines; $A = 1$, $\alpha = 3$.

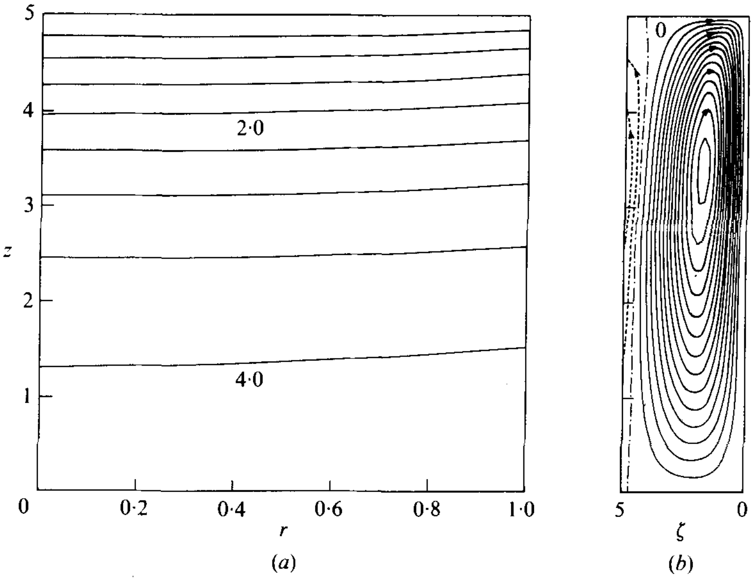

FIGURE 5. (a) Isotherms in the inner core. (b) Streamlines in the side-wall Stewartson $E^{\frac{1}{3}}$-layer. The difference in the values of the non-dimensional flux on adjacent lines is $-0{\cdot}03$ for solid lines and $0{\cdot}01$ for dashed lines; $A = 5$, $\alpha = 0{\cdot}3$.

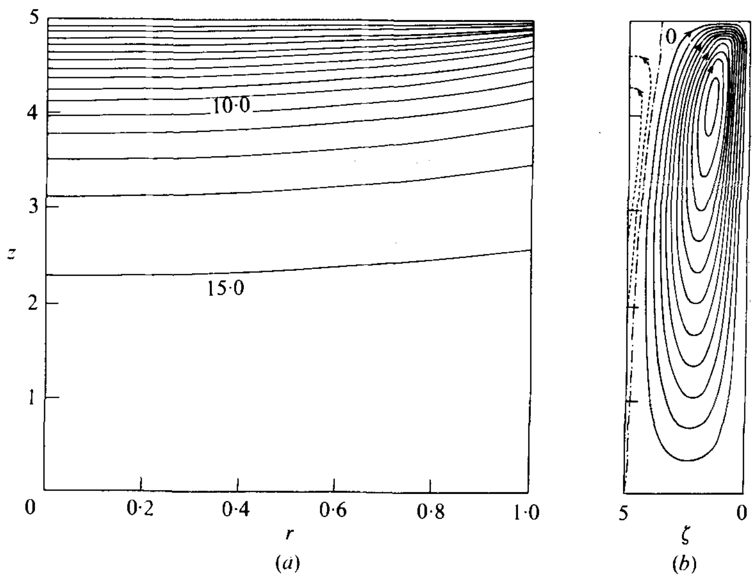

FIGURE 6. (a) Isotherms in the inner core. (b) Streamlines in the side-wall Stewartson $E^{\frac{1}{3}}$-layer. The difference in the values of the non-dimensional flux on adjacent lines is $-0{\cdot}01$ for solid lines and $0{\cdot}005$ for dashed lines; $A = 5$, $\alpha = 3$.

The infinite linear algebraic equations are truncated and solved by Gauss's sweepout method. Retaining the first 80 terms only is found to give sufficient accuracy and convergence.

The results of the calculation are summarized in figures 3–6. (Note that only the upper half of the cylinder is shown.) The effects of compressibility can be seen from a comparison of figures 3 and 4 and of figures 5 and 6. For fixed A, both the induced temperature in the inner core and the flux of the closed circulation in the $E^{\frac{1}{3}}$-layer become larger for larger α. Comparison of figures 3 and 5 and of figures 4 and 6 shows that, for fixed α (the induced temperature), the flux and also the thickness of the regions of the closed circulation become larger for larger A. The reason for this variation with varying A is that the layer of heat production (near the side wall) becomes larger compared with the heat-loss area (near the end plates).

5. Conclusion and discussion

The results of the calculations reported here have shown that the flow depends crucially on the thermal nature of the side wall. If the side wall is thermally conducting, i.e. its temperature is fixed, then the effect of the wall is restricted to the narrow Stewartson $E^{\frac{1}{3}}$- and $E^{\frac{1}{4}}$-layers, this situation being the same as in the case of Boussinesq fluids. However, if the side wall is thermally insulated, the situation is much more complicated. We have studied the case in which $\alpha\ (=\frac{1}{4}(\gamma-1)P_r G_0 E^{-\frac{1}{3}})$ is $O(1)$. In this case, a meridional flow in the $E^{\frac{1}{4}}$-layer, induced by Ekman suction, produces a perturbation in the temperature $O(E^{\frac{1}{12}})$ in the inner core. The temperature perturbation produces a thermal wind $O(E^{\frac{1}{12}})$ in the inner core, and this thermal wind induces a closed circulation $O(E^{\frac{5}{12}})$ in the $E^{\frac{1}{3}}$-layer. The resultant inner flow is dragged by the side wall, with a velocity of the order of $E^{\frac{1}{12}}$. Such a dragging phenomenon does not appear in Boussinesq fluids, nor in compressible fluids confined in a cylinder with a thermally conducting side wall.

The present result would be useful in constructing the type of gas centrifuge proposed by Matsuda (1976*b*), in which the end plates rotate faster than the side wall. In this type of centrifuge the inner core should not be dragged very much by the side wall because dragging reduces the efficiency of separation of isotopes. In order to avoid dragging therefore, we have to keep the temperature of the side wall constant by some means. In this case the dragging would be at most of the order h, and since we can reduce h significantly by adding light gas such as H_2 or He to the UF_6, such dragging would be negligible.

The present analysis is also useful from another practical viewpoint. As Durivault *et al.* (1976) pointed out, a strong closed circulation within the Stewartson layer is harmful for the separation because of mixing loss. In order to prevent the circulation in the $E^{\frac{1}{3}}$-layer, they suggested the use of either an isothermal side wall or an insulated side wall, the latter possibility being derived from the analysis of III. However, as shown here, in the case in which an $E^{\frac{1}{4}}$-layer exists at the side wall, an insulated wall does not suppress the closed circulation in the $E^{\frac{1}{3}}$-layer but enhances it. Therefore it would be better to use an isothermal side wall, if at all practical from an engineering viewpoint, rather than an insulated one.

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